# Mono-anabelian Reconstruction of Number Fields


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RIMS


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gp = group

fct = function

alg'm = algorithm

\mathcal $\mathcal{D}$ $\mathcal{F}$

\mathfrak $\mathfrak{S}$ $\mathfrak{I}$

(Hodge ⇝ p-adic Hodge ⇝ ) Hodge-Arakelov theory

§

application

(absolute) anabelian geometry ① inter-universal of $\Theta$-link

③ motivation

geometry (logically no need)

④ application

② philosophical view point

(complex / p-adic) Teichmüller theory Diophantine inequality

gp = group

fct = function

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(absolute) anabelian geometry

application

① inter-universal geometry

③ motivation of Θ-link

(logically no need)

④ application

② philosophical view point

(complex / p-adic) Teichmüller theory

Diophantine inequality

① ~ ④ Mochizuki's discoveries

gp = group

\mathcal $\mathcal{D}$ $\mathcal{F}$

① anab $\rightsquigarrow$ int-univ.

**Galois theory**: $F$: field

$$G_F \supset \begin{matrix} G_{K_1} \\ \parallel \\ G_{K_2} \end{matrix} \Rightarrow K_1 = K_2$$

**anabelian geom**

Neukirch-Uchida $K_1, K_2 / \mathbb{Q}$ fin.

$$\underset{\text{as top. gp}}{G_{K_1} \cong G_{K_2}} \Rightarrow \underset{\text{as fields}}{K_1 \cong K_2}$$

$X, X$ : hyperbol. curve / field

$X_1, X_2$: hyperb. curves / field $F$

Under some conditions

outer isoms $\downarrow$ arith. fund. $\curvearrowleft$ gp

$$\mathrm{Isom}_F(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}^{\mathrm{out}}_{G_F}(\Pi_{X_1}, \Pi_{X_2})$$

(Hom ...)

Q. When do we treat $\Pi_X$ or $G_F$ as abstract top. gps?

A. In IUTch !!

(possibly first time)

$X_1, X_2$: hyperb. curves / field $F$

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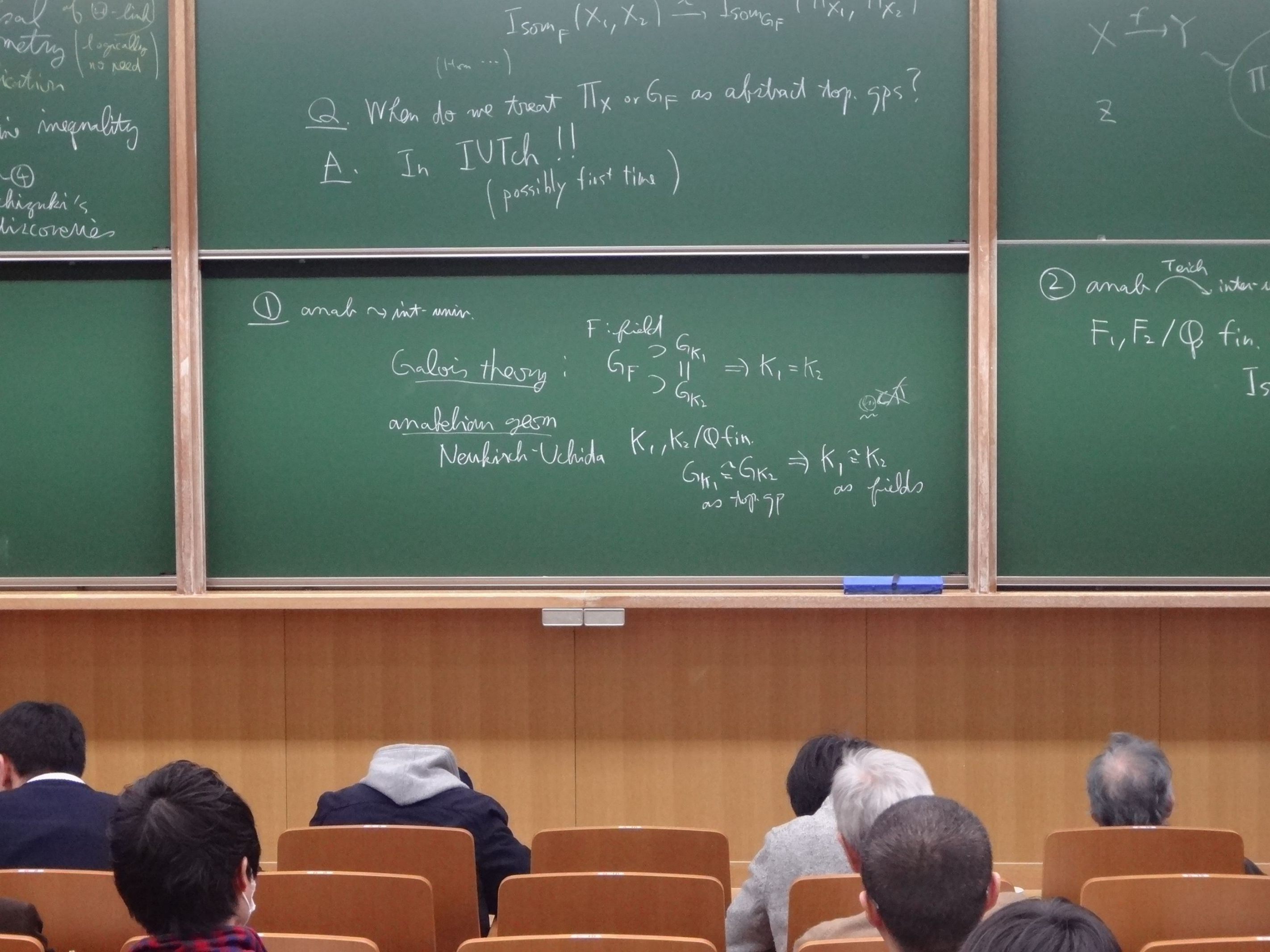


(logically no need)
inequality
Mochizuki's discoveries
$\mathrm{Isom}_F(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}_{G_F}(\Pi_{X_1}, \Pi_{X_2})$
(Hom ···)
Q. When do we treat $\Pi_X$ or $G_F$ as abstract top. gps?
A. In IUTch !!
(possibly first time)
$X \xrightarrow{f} Y$
$Z$
① anab ⇝ int-univ.
$F$: field
Galois theory : $G_F \supset G_{K_1} = G_{K_2} \Rightarrow K_1 = K_2$
anabelian geom
Neukirch-Uchida $K_1, K_2/\mathbb{Q}$ fin.
$G_{K_1} \cong G_{K_2} \Rightarrow K_1 \cong K_2$
as top. gp
as fields
② anab $\xrightarrow{\text{Teich}}$ inter-u
$F_1, F_2/\mathbb{Q}$ fin.

anab. fund.

$\Pi_{X_2}$ )

gps?

$K_2$

fields

inter-universality

$X \xrightarrow{f} Y$

$Z$

$\Pi_X \xrightarrow{f_*} \Pi_Y \cong \Pi_Z$

as abs. top. gps

in general

this arrow does not come from scheme theory

"Teichmüller dilation"

need to distinguish

labeling systems

scheme theories

$\mathbb{C} \cong \mathbb{R}^2$

② anab $\overset{\text{Teich}}{\rightsquigarrow}$ inter-univ.

$F_1, F_2 / \mathbb{Q}$ fin.

$\mathrm{Isom}(\overline{F}_1/F_1, \overline{F}_2/F_2)$

$\subset \mathrm{Isom}(Gal(F$

(2) anab. $\overset{\text{Teich}}{\curvearrowright}$ inter-univ.

$F_1, F_2 / \mathbb{Q}$ fin.

$$\mathrm{Isom}(\overline{F}_1/F_1, \overline{F}_2/F_2) \subsetneq \mathrm{Isom}(\mathrm{Gal}(\overline{F}_2/F_2), \mathrm{Gal}(\overline{F}_1/F_1))$$

cf. [NSW] Chap. VII §5 P.420–423

$$\mathrm{Aut}(F) \subsetneq \mathrm{Out}(G_F)$$

$\exists$ outer autom's which do not come from isom's of (top.) fields

inter-universality

$X \xrightarrow{f} Y$

$\Pi_X \xrightarrow{f_*} \Pi_Y \cong \Pi_Z$

as abs. top. gps

need to distinguish

labeling systems

scheme theories

$\mathrm{codim}(G_F) = 2$

$$1 \to \underset{\sim\sim}{I_F} \to G_F \to \hat{\mathbb{Z}}\,\underset{\sim\sim}{\mathrm{Frob}_F} \to 1$$

$\exists$ autom's which do not come from field theory

inter-universally non-rigid

can recover gp th'cally

inter-universally rigid

LCFT

$$0 \to \mathcal{O}_F^\times \to F^\times \xrightarrow{\mathrm{val}} \mathbb{Z} \to 0$$

LCFT

$$0 \longrightarrow \mathcal{O}_F^\times \longrightarrow F^\times \xrightarrow{\text{val}} \mathbb{Z} \longrightarrow 0$$

non-rigid

rigid

$F:$ $\boxplus$ $\boxtimes$

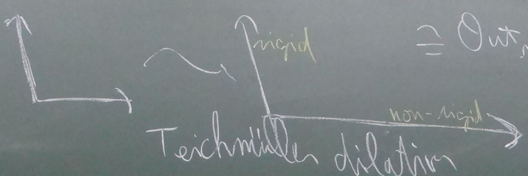
$G_F \rightsquigarrow F^\times$

can recover only $\boxtimes$

cannot recover $\boxplus$

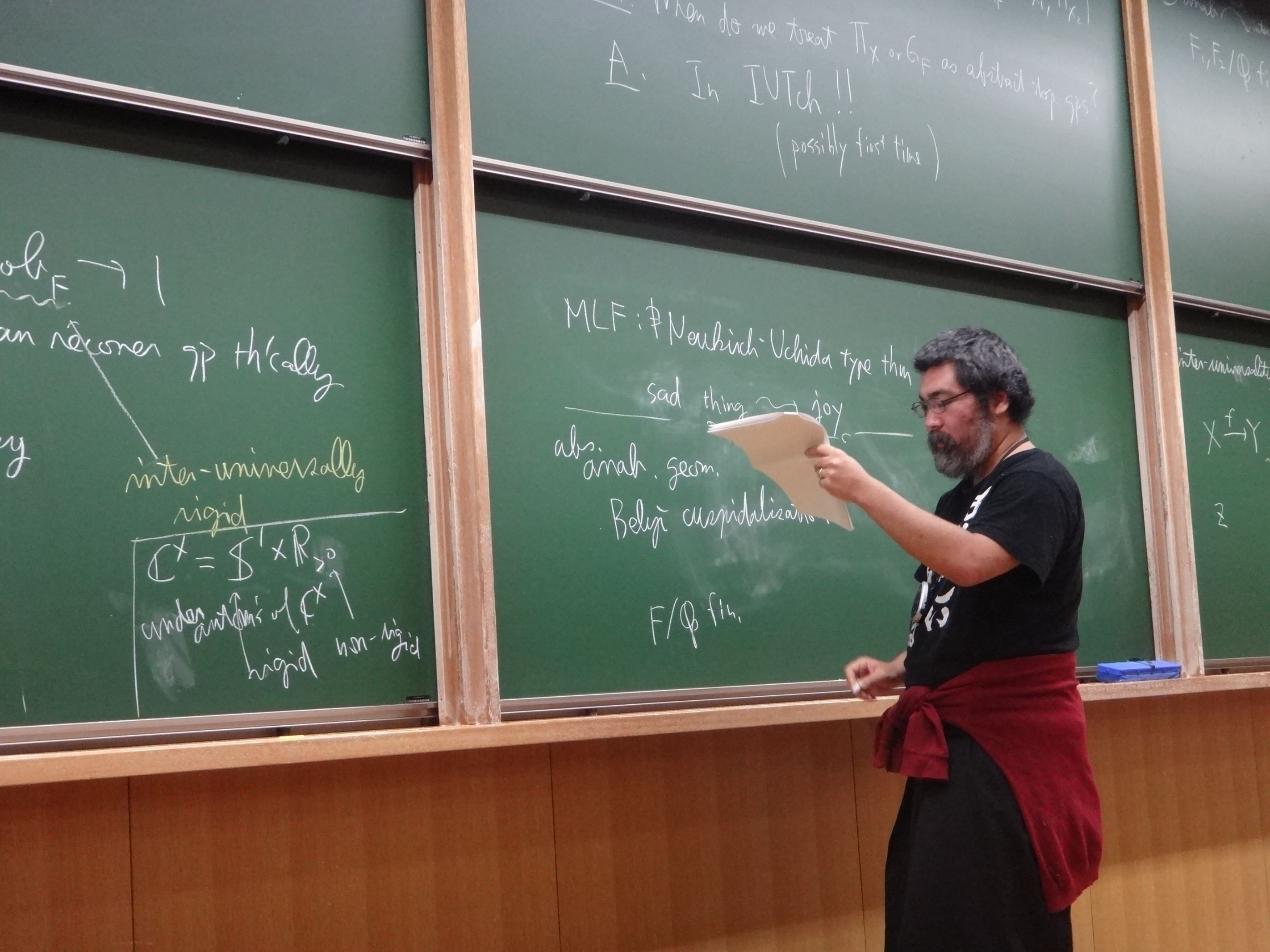
cf. $\mathrm{Isom}(F_1, F_2) \cong \mathrm{Out}_{\text{ram. fil}}(G_{F_2}, G_{F_1})$

[pGC]




$\mathrm{codim}(G_F) = 2$

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \hat{\mathbb{Z}}\,\mathrm{Frob}_F \longrightarrow 1$$



When do we treat $\Pi_X$ or $G_F$ as abstract top. gps?
A. In IUTch !!
(possibly first time)
gp th'cally
inter-universally rigid
$\mathbb{C}^\times = S^1 \times \mathbb{R}_{>0}$
rigid non-rigid
MLF: ∄ Neukirch-Uchida type thm
sad thing ⇝ joy
abs. anab. geom.
Belyi cuspidalization
$F/\mathbb{Q}$ fin.
$X \xrightarrow{f} Y$

MLF : $\not\exists$ Neukirch-Uchida type thm

sad thing $\rightsquigarrow$ joy

abs. anal. geom.

$G_F \overset{\text{alg'm}}{\rightsquigarrow}$ mult. gp $F^\times$

Belyi cuspidalization

$X$ : strictly Belyi type $/F$

$F/\mathbb{Q}$ fin.

$\Pi_X \rightsquigarrow$ top. field $F$

$X_1, X_2$ : hyperb. curves / field $F$

$, G_{F_1})$

$\mathbb{C}]$

$G_F$ , $G_F \curvearrowright \mathcal{O}_F^{\triangleright} := \mathcal{O}_F \setminus \{0\}$ "mono-analytic" ($\rightsquigarrow$ real analytic)

$\Pi_X$ , $\Pi_X \curvearrowright \mathcal{O}_F^{\triangleright}$ "arithmetically holomorphic" ($\rightsquigarrow$ holomorphic)

regard $\Pi_X$ as "tangent vector" of $F$

$\rightsquigarrow$ rigidify

the non-rigid dimension

② anab. Teich. inter-univ.

$F_1, F_2 / \mathbb{Q}_p$ fin

$\mathrm{Isom}(\overline{F}_1/F_1, \overline{F}$

$\mathrm{Aut}(F)$

$\subsetneq \mathrm{Out}$

inter-universality

MLF: $\nexists$ Neukirch-Uchida type thm

sad thing $\rightsquigarrow$ joy

$F^\times$

$X \xrightarrow{f} Y$

$\Pi_X \xrightarrow{f_*} \Pi$

③ Hodge-Arakelov $\rightsquigarrow$ inter-univ.

p-adic Hodge

Hodge

Thm of dR / $\mathbb{C}$

$$H_1(\mathbb{C}^\times, \mathbb{Z}) \otimes H^1_{dR}(\mathbb{C}^\times) \longrightarrow \mathbb{C}$$

$$\circlearrowleft \otimes \frac{dT}{T} \longmapsto \int_{\circlearrowleft} \frac{dT}{T} = 2\pi i$$

perfect pairing

p-adic Hodge / $\mathbb{Q}_p$

$T_p \mathbb{G}_m$

p-adic Hodge / $\mathbb{Q}_p$

$$T_p \mathbb{G}_m \otimes H^1_{dR}(\mathbb{G}_m/\mathbb{Q}_p) \longrightarrow B_{crys}$$

p-adic Tate module $\|$

$$\varprojlim_n \mathbb{G}_m[p^n](\overline{\mathbb{Q}}_p) \ni \underline{\varepsilon} \otimes \frac{dT}{T} \longmapsto \text{"}\int_{\underline{\varepsilon}} \frac{dT}{T}\text{"} = \log[\underline{\varepsilon}] = t \updownarrow (2\pi i)$$

$$\underline{\varepsilon} = (\varepsilon_n)_n$$

$$\varepsilon_1 = 1,\ \varepsilon_2 \neq,\ \varepsilon_{n+1}^p = \varepsilon_n$$

"analytic path"

③ Hodge-Arakelov $\rightsquigarrow$ inter-univ.

$\sim$ p-adic Hodge

Thm of dR / $\mathbb{C}$

$$H_1(\mathbb{C}^\times, \mathbb{Z}) \otimes H^1_{dR}(\mathbb{C}^\times) \longrightarrow \mathbb{C}$$

ys

$= \log[\underline{\varepsilon}]$

$= \star$

$(2\pi i)$

th"

$E / \mathbb{Z}_p$ ell. curve

$$T_p E \otimes \underbrace{H^1_{dR}(E/\mathbb{Q}_p)}_{\substack{\| \\ \mathrm{coLie}\, E^\dagger}} \longrightarrow B_{crys}$$

$$\underline{P} \otimes \omega \longmapsto \text{"} \int_{\underline{P}} \omega \text{"} \in$$

$$\underline{P} = (P_n)_n \quad P_1 = 0,\ pP_{n+1} = P_n \qquad \text{"} \log_\omega([\underline{P}]) \text{"}$$

"analytic path"

$$\log_\omega : \hat{E}_{/\mathbb{Q}_p} \xrightarrow{\ \sim\ } \hat{\mathbb{G}}_{a,\mathbb{Q}_p}$$

$$(\log_\omega)^* dT = \omega$$

univ. ext'n

$$0 \to \mathrm{coLie}\, E^\vee \to E^\dagger \to E \to 0$$

$/\mathbb{C}$ Hodge fil $H_1(E,\mathbb{Z}) = H_1(E,\mathbb{Z})$

$$0 \to \mathrm{coLie}\, E^\vee \to \mathrm{Lie}\, E^\dagger \to \mathrm{Lie}\, E \to 0$$

$\|$ $\quad (H^1_{dR})^*$

$$0 \to \mathrm{coLie}\, E^\vee \to E^\dagger \to E \to 0$$

0

# Hodge-Arakelov

$F$ : NF

$H^1_{dR}(\mathbb{G}_m/\mathbb{Q}_p) \longrightarrow B_{crys}$

dR

$\otimes \frac{dT}{T} \longmapsto$ "$\int_{\underline{\varepsilon}} \frac{dT}{T}$" $= \log[\underline{\varepsilon}] = \star$ $(2\pi i)$

$, \varepsilon_{n+1}^p = \varepsilon_n$

"analytic path"

## Hodge-Arakelov

$F$: NF, $E/F$ ell. curve, $l > 2$ prime

$E(F)[2] \ni P \neq 0$, $\mathcal{L} := \mathcal{O}(l[P])$ line bdl over $E$ $\dim_F = l^2$

dR side

étale side!

Roughly $\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg < l} \xrightarrow{\sim} \mathcal{L}|_{E^\dagger[l]} = \mathcal{L}|_{E[l]} = \bigoplus_{E[l]} F$

$\dim_F = l^2$

$E^\dagger$ in Zar. locally $\cong \mathbb{G}_a/E = \mathrm{Spec}\, \mathcal{O}_E[T]$

rel. deg

isom of F-vect. space & preserves specified (Arch / non-Arch) integral str.

ir.

$) \otimes H^1_{dR}(\mathbb{C}^\times) \longrightarrow \mathbb{C}$

$\otimes \frac{dT}{T} \longmapsto \int_{C} \frac{dT}{T} = 2\pi i$

perfect pairing

$E/\mathbb{Z}_p$ ell. curve

$T_p E \otimes H^1_{dR}(E/\mathbb{Q}_p) \longrightarrow B_{crys}$

et, dR, $= \omega_{\mathrm{Lie}\, E^\dagger}$

$\underline{P} \otimes \omega \longmapsto$ "$\int_{\underline{P}} \omega$" = "$\log_\omega([\underline{P}])$"

"analytic path"

$\underline{P} = (P_n)_n$, $P_1 = 0$, $pP_{n+1} = P_n$

$\log_\omega : \hat{E}_{\mathbb{Q}_p} \xrightarrow{\sim} \hat{\mathbb{G}}_{a,\mathbb{Q}_p}$

$(\log_\omega)^* dT = \omega$

univ. ext'n

$0 \to \omega_{E^\vee} \to E^\dagger \to E \to 0$

/$\mathbb{C}$ Hodge fil $H_1(E,\mathbb{Z}) = H_1(E,\mathbb{Z})$

$0 \to \omega_{E^\vee} \to \mathrm{Lie}\, E^\dagger \to \mathrm{Lie}\, E \to 0$

$0 \to \omega_{E^\vee} \to E^\dagger \to E \to 0$

We regard

$E[\ell]$ as an "approximation of underlying mfd of $E$"

(feeling $\ell \gg 0$ cf. IUT $\ell$ "$\approx$" $ht(E)$)

(consider $\mathcal{L}|_{E[\ell]}$, not $E[\ell]$ itself

"fcts on $E[\ell]$"

$\rightsquigarrow$ quantisation)

cf. degenerate case $\mathbb{G}_m$ case

$$\begin{array}{ccc} F[T]^{\deg<\ell} & \xrightarrow{\sim} & \bigoplus_{\xi\in\mu_\ell} F \\ \Downarrow & & \Downarrow \\ f & \longmapsto & (f(\xi))_{\xi\in\mu_\ell} \end{array}$$

Vandermonde det $\neq 0$

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg<\ell} \xrightarrow{\sim} \mathcal{L}|_{E[\ell]}$$

fil by rel. deg ... $^{-j} \cong$

$E$ $\dim_F = \ell^2$

étale scheme

$= \bigoplus F$

$E[\ell]$ $E[\ell]$

$\mathcal{O}_E[T]$

rel. deg

int. str.

ext'n

$E \to 0$

$H_1(E,\mathbb{Z})$

$E^\dagger \to \mathrm{Lie}\,E \to 0$

---

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg < \ell} \xrightarrow{\sim} \mathcal{L}|_{E[\ell]}$$

$\begin{pmatrix} \omega^{\otimes 2} \simeq \Omega \\ \mathbb{Q}^{\times}\ 2\pi i \\ \omega \mapsto \sqrt{\pi} \end{pmatrix}$

fil by rel. deg. $\quad gr^{-j} \simeq (\mathrm{coLie}\,E)^{\otimes(-j)}$ $\quad$ We consider on the moduli

$\mathrm{coLie}\,E = \omega_E$ $\quad$ int. str. Gaussian pole $\quad$ of ell. curves

+Lefschetz fil & its derivative $\quad \omega_E \longleftarrow$ theta values $\quad$ polar coord. $\quad q^{j^2/8\ell}\,\mathcal{O}_F$ $\quad \mathcal{M}_{ell}$

$$\deg \text{ of LHS} \quad -\sum_{j=0}^{\ell-1} j\,[\omega_E] \overset{\deg}{\approx} -\frac{\ell^2}{2}[\omega_E]$$

q-parameter at bad prime

$$= -\frac{\ell^2}{24}[\log q] \qquad \qquad [\omega_E^{\otimes 2}] \overset{KS}{=} [\Omega_{\mathcal{M}_{ell}}]$$

$$\deg \text{ of RHS} \quad -\frac{1}{8\ell}\sum_{j=0}^{\ell-1} j^2 [\log q] \approx -\frac{\ell^2}{24}[\log q] \qquad \qquad = \frac{1}{6}[\log q]$$

Gaussian distribution $\leftarrow$ cartesian coord.

We regard

$E[\ell]$ as an "approximation of underlying mfd of $E$"

(feeling $\ell >> 0$ cf. IUT $\ell$ "$\approx$" $ht(E)$) $\quad$ (consider $\mathcal{L}|_{E[\ell]}$, not $E[\ell]$ itself

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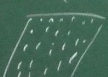


cf. degenerate case $\mathbb{G}_m$ case

$$F[T]^{\deg < \ell} \xrightarrow{\sim} \bigoplus_{\zeta \in \mu_\ell} F \quad \longleftarrow \text{Vandermonde det} \neq 0$$

$$f \mapsto (f(\zeta))$$

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$\omega_E$

cart. coord.

$\omega_E^{\otimes 2} = \Omega_{M_{ell}}(\log)$

polar coord.

$\int C = 2\pi i$

- anabelian geom $\leftarrow$ tools
- Hodge-Arakelov $\leftarrow$ "design" "story"

$E[l]$

$E[\ell] \supset M$ ← rank = 1, at non-arch bad $\ni \mu_\ell$

$\not\exists$ gl. mult. sub (in general)

<u>If it existed</u>

$K := F(E[\ell])$, $E' := E/M$

apply Hodge-Arakelov to

$q = (q_v)_{v: \text{bad}}$ q-parameter

$$\Gamma\left((E')^\dagger, \mathcal{L}|_{(E')^\dagger}\right)^{\deg < \ell} \longrightarrow \bigoplus_{-\frac{\ell-1}{2} \leq j \leq \frac{\ell-1}{2}} \left(q^{\frac{j^2}{2\ell}} \mathcal{O}_K\right) \otimes_{\mathcal{O}_K} K$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

in LHS

Hodge fil is incompat. w/ direct sum decomp. in RHS

$\rightsquigarrow$ proj. to the $j$-th factor $\leftarrow$ $j \approx l \approx ht(E)$

$$\mathrm{Fil}^0 = \underline{\underline{q}}\, \mathcal{O}_K \underset{\text{non-trivial}}{\hookrightarrow} \underline{\underline{q}}^{j^2} \mathcal{O}_K$$

$\underline{\underline{q}} := q^{1/2l}$

arith Kodaira-Spencer map

$\left(\begin{array}{l}\deg \doteqdot 0 \\ \text{arith line} \\ \text{bdl}\end{array}\right)$ $\left(\begin{array}{l}\deg \ll 0 \\ \text{arith line} \\ \text{bdl}\end{array}\right)$

$\rightsquigarrow 0 \lesssim -(\text{large number})$ $(\approx$

$\rightsquigarrow \quad 0 \lesssim -(\text{large number})\ (\approx - ht(E))$

$\rightsquigarrow \quad ht \lesssim 0$ ht is bounded

<u>Want</u> $\underline{q}\,\mathcal{O}_K \hookrightarrow \underline{q}^{j^2}\mathcal{O}_K$ (*)

Hodge-Arakelov : use scheme theory, cannot obtain (*)

IVTch : use (*), abandon scheme theory

Hodge fil in LHS is incompat. w/ direct sum decomp. in RHS

$\rightsquigarrow$ proj. to the $j$-th factor $\leftarrow j \approx l \approx ht(E)$

...:bad
...-parameter
$K$
$\mathcal{O}_K$

$\Theta(E^{\dagger})$
fil by rel. deg
theta ... & ...
...
(H)-link

$- ht(E))$

ounded

cannot obtain

$(*)$

andom

scheme theory

## Θ-link

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{1\le j\le \frac{l-1}{2}}^{\mathbb{N}} \longmapsto \underline{\underline{q}}^{\mathbb{N}}$$

$$\underline{q}^{j^2}\mathcal{O}_K \longleftrightarrow \underline{q}\,\mathcal{O}_K$$

"identify"

non-scheme theoretic link

Θ-link is almost* "tautological solution of Diophantine inequality"

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\deg < l} \xrightarrow{\sim} \mathcal{L}|_{E[l]}$$

$$\left(\begin{array}{l} \omega^{\otimes 2} \simeq \Omega \\ \quad \uparrow 2\pi i \\ \omega \mapsto \sqrt{\pi} \end{array}\right)$$

fil by rel. deg. $\quad gr^{-j} \simeq (\underbrace{\mathrm{coLie}\,E}_{\omega_E})^{\otimes(-j)}$

n RHS

We conside[…] on the moduli

$j \approx l \approx ht(E)$

int. str. Ga[…] of ell. curves $\overline{\mathcal{M}}_{ell}$

theta fct & its derivative $\quad \omega_E \longleftarrow$ theta values

polar[…] coo[…] F

[…]arameter at bad pr[…]

deg of LHS $\quad -\sum_{j=0}^{l-1} j[\omega_E] \approx -\frac{l^2}{2}[\omega_E]$

$= -\frac{l^2}{2[…]}$ $\quad [\omega_E^{\otimes 2}] \overset{KS}{=} [\Omega_{\overline{\mathcal{M}}_{ell}}^{\log}]$

Spencer map

deg of RHS $\quad -\frac{1}{8l}\sum_{j=0}^{l-1} j^2[\log q] \approx -\frac{l^2}{24}[\log […]$

$= \frac{1}{6}[\log q]$

Gaussian distribut[…] […]n cond.

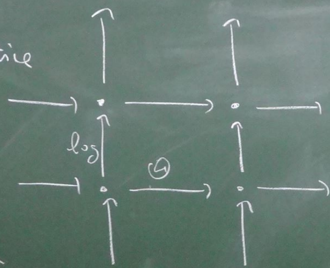
lie bdd

coord.

"design"
"story"

④ inter-univ ⇝ Dioph. ineq.

log-Θ-lattice




• : Hodge theater

Θ-link

$\{ q^{j^2} \}_{1 \le j \le \frac{\ell-1}{2}}$ ( non-scheme-theoretic ) $q$

ht fct ← X → ht fct

scheme theory | scheme theory

$2\Theta = 6$

Θ-link $2x = 6$

analytic ineq. $x = 3$

$X, Y$ / $k$: curves

$X^{top} \cong Y^{top}$ Zariski homeo

Θ

$\uparrow$ : log-link

$\rightarrow$ : $\Theta$-link

p n h k

| pTeich | | IUTch |
| --- | --- | --- |
| hyperb. curve / char > 0 | $\longleftrightarrow$ | NF |
| indig. bdl | $\longleftrightarrow$ | once punctured ell. curve / NF |
| Frob. in char > 0 | $\longleftrightarrow$ | $\uparrow$ log-link |
| lift $p^n/p^{n+1}$ $\rightsquigarrow$ $p^{n+1}/p^{n+2}$ in Witt ring | $\longleftrightarrow$ | $\overrightarrow{\Theta}$-link |

| | | |
| --- | --- | --- |
| can. Frob. lift | $\longleftrightarrow$ | log-$\Theta$ lattice |

④ inter-univ $\rightsquigarrow$ Dioph. ineq.

log-Θ-lattice

$$
\begin{tikzcd}
 & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & {} \\
{} \arrow[r] & \bullet \arrow[u, "\log"] \arrow[r, "\Theta"] & \bullet \arrow[u] \arrow[r] & {}
\end{tikzcd}
$$

•: Hodge theater

alien ring str. ⟵ eye

Want to recover from the universe in the right hand side (under mild indet.)

Θ-link

non-scheme

$$\{q^{j^2}\}_{1 \le j \le \frac{\ell-1}{2}} \xleftrightarrow{\text{theorem}} \underline{\underline{q}}$$

ht fct ⟵ X ⟶ ht fct

scheme theory | scheme theory

Θ-link {analytic inequality

$2\Theta = 6$

$2x = 6$

$\vdots$

$x = 3$

$X, Y$ / $\bar{\mathbb{F}}$ curves

$X^{(p)} \cong Y^{(p)}$ Zariski homeo

Θ

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

1. anabelian geom

$2 ① = 6$

$2x = 6$

$\downarrow \div$

$x = 3$

$X, Y \ /k = \bar{k}$ curves

$X^{t\,\mathrm{p}} \cong Y^{t\,\mathrm{p}}$ Zariski hom.

$\Theta$

dichotomy
| Frob.-like obj. : abstract top. monoids etc. order respect
| étale-like obj. : $\Pi_X$, $G_F$ & obj. reconstructed from $\Pi_X$, $G_F$

via anabelian recon. alg'm

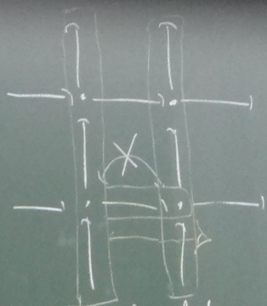
$^{\dagger}\Pi_X \rightsquigarrow \mathcal{O}^{\triangleright}(^{\dagger}\Pi_X) \rightsquigarrow M \ (\cong \mathcal{O}^{\triangleright}(^{\dagger}\Pi_X))$

étale-like — abs. top. monoid, Frob.-like

indifferent to order

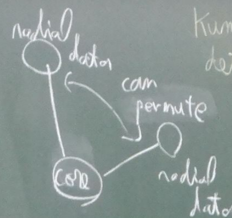
to penetrate the walls

| | pTeich | IUTch | | |
|---|---|---|---|---|
| $\uparrow$ : log. link | hyperb. curve/char $p$ $\longleftrightarrow$ | NF | can. Frob. lift $\longleftrightarrow$ | log-$\Theta$ lattice |

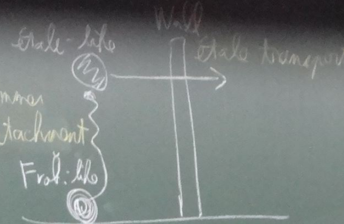


Frob.-like picture

Kummer detachment



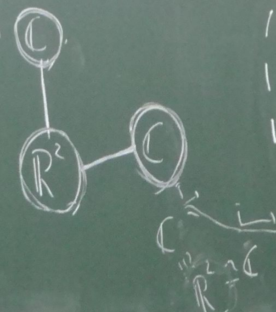

étale-like picture

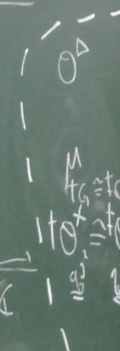



cart. coord.

polar coord.

cf.




<u>main thm of IUTch</u> ([IUTch III, Th3.11])

main thm of IUTch ([IUTch III, Th 3.11])

After admitting 3 indet. $(\mathrm{Indet}\overset{2}{\rightarrow})$, $(\mathrm{Indet}\overset{3}{\uparrow})$, $(\mathrm{Indet}\overset{1}{\circlearrowright})$,

Θ-link → Kummer detach.

log-link ↑

ét. transp.

we can put both sides of Θ-link $\{\underline{\underline{q}}^{j^2}\}_{1\leq j\leq \frac{\ell-1}{2}} \longleftrightarrow \underline{\underline{q}}$

on the same platform & we cannot distinguish them.

⇝ can compare deg of line bdles on the both sides (under mild indet)

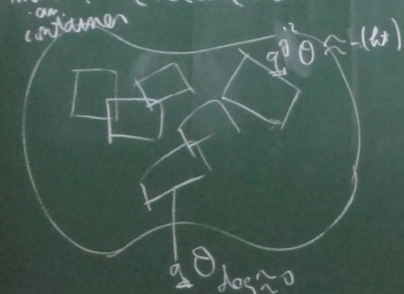
conseq.

these indet. are mild ones.

(↑ deep & need delicate constructions)

↑

calculate concretely ([IUTchIV])




$$0 \lesssim -(ht) + \underbrace{(\text{indet.})}_{= \text{log-diff } (+\text{ log.-cond.})}$$

$\rightsquigarrow$ $\underline{ht \lesssim \text{log-diff.} + \text{log.-cond.}}$ Vojta's inequality

tomorrow
9:30-11:00
14:00-18:00

M. Kim
Math. Faculty
Building 3
Room 127
11:20 - 12:20

## ingredients

([pGC] ⇝ [AbsTopI])
§4
[AbsTopII]
§3
[AbsTopIII]

① Belyi & elliptic cuspidalizations [AbsTopIII] §1,2
(& its Arch. analogue) (w/ [AbsAnab], [CanLift], [AbsSect], [CombGC] [Corn], [Cusp])

② mono-theta env.
3 rigidities [EtTh] §1,2

$\Pi \rightarrow \mathbb{M} \leftarrow \bar{\mathcal{F}}$

③ Frd theoretic mono-theta env. [EtTh] §3~5
(Frd [FrdI], [FrdII])

④ semi-graphs of anabelioids [SemiAnab]
(~ [Anab]

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## ingredients

[ [pGC] ~> ] [AbsTop I]
↓ §4
[AbsTop II]
↓ §3
[AbsTop III]

① Belyi & elliptic cuspidalizations [AbsTopIII] §1,2
(& its Arch. analogue)
(w/ [AbsAnab], [CanLift], [AbsSect], [CombGC], [Corr], [Cusp])

② mono-theta env.
3 rigidities [EtTh] §1,2

③ Fr'd theoretic mono-theta env. [EtTh] §3-5
( Fr'd [FrdI], [FrdII] )

$\Pi \to \mathbb{M} \leftarrow \mathcal{F}$

④ semi-graphs of anabelioids [SemiAnab]
( ~ [Anab] )

⑤ log-shells [AbsTopIII] §3,4

mod. conj. vs temp. conj.

⑥ reductions by usual arith. geom.
[GenEll] ~ [Belyi]

loc. temp. Δ
gl. Δ prof.

$\int e^{-x^2} dx$

cf. [Pano]
[HASurI], [HASurII]

main
Aft
we can

## §2. Preliminaries on Anabelian Geom.

### §2.1 Some Basics on Galois Group of local fields

Prop 2.1 $i=1,2$, $K_i / \mathbb{Q}_{p_i}$ fin. w/ res. field $k_i$

$\overline{K}_i$ : alg. closure of $K_i$ w/ res. field $\overline{k}_i$

$e(K_i)$ : ram. index of $K_i$ over $\mathbb{Q}_{p_i}$

$f(K_i) := [k_i : \mathbb{F}_{p_i}]$, Put $G_{K_i} := \mathrm{Gal}(\overline{K}_i / K_i)$

$(G_{K_i} \supset) I_{K_i}$ : inertia $\supset P_{K_i}$ : wild inertia

$\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$ isom. of prof. gps

(1). $p_1 = p_2 \ (=: p)$

(2). $\alpha^{ab} : G_{K_1}^{ab} \xrightarrow{\sim} G_{K_2}^{ab}$ & $k_1^\times \subset \mathcal{O}_{K_1}^\times \subset K_1^\times \subset G_{K_1}^{ab}$ (LCFT)

induce isom's

(a). $\alpha^{ab} : k_1^\times \xrightarrow{\sim} k_2^\times$

(b)

[Top I] §4 [Top II] §3 [Top III] 1,2 [Can Lift], [Comb GC] [Cusp]

$\Pi$, $\overline{X}$, $\mathbb{M}$

[Anab] conj. temp. conj. loc. temp $\Delta$ temp.

there is

mono-an containers calc

## §2. Preliminaries on Anabelian Geom.

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(1). $p_1 = p_2 \ (=: p)$

(2). $\alpha^{ab} : G^{ab}_{K_1} \xrightarrow{\sim} G^{ab}_{K_2}$ & $k_i^\times \subset \mathcal{O}^\times_{K_i} \subset K_i^\times \overset{\text{LCFT}}{\subset} G^{ab}_{K_i}$

induce isom's

(a). $\alpha^{ab} : k_1^\times \xrightarrow{\sim} k_2^\times$

(b). $\alpha^{ab} : \mathcal{O}^\times_{K_1} \xrightarrow{\sim} \mathcal{O}^\times_{K_2}$

(c). $\alpha^{ab} : \mathcal{O}^\triangleright_{K_1} \xrightarrow{\sim} \mathcal{O}^\triangleright_{K_2}$

(d). $\alpha^{ab} : K_1^\times \longrightarrow K_2^\times$

(3). (a). $[K_1 : \mathbb{Q}_p] = [K_2$

(b). $f(K_1) = f(K_2$

(c). $e(K_1) = e(K_2$

(4) the restriction of $\alpha$

(a). $\alpha|_{I_{K_1}} : I_{K_1}$

(b). $\alpha|_{P_{K_1}} : P_{K_1}$

(5) the induced map

(6). the collection of the

induces Mor(

which is

(7) the isom $\alpha^{\times}$

induced by

(3). (a). $[K_1 : \mathbb{Q}_p] = [K_2 : \mathbb{Q}_p]$

(b). $f(K_1) = f(K_2)$

(c) $e(K_1) = e(K_2)$

(4) the restrictions of $\alpha$ induce

(a) $\alpha|_{I_{K_1}} : I_{K_1} \xrightarrow{\sim} I_{K_2}$

(b) $\alpha|_{P_{K_1}} : P_{K_1} \xrightarrow{\sim} P_{K_2}$

(5) the induced map $G^{ab}_{K_1}/\mathrm{Im}(I_{K_1}) \xrightarrow{\sim} G^{ab}_{K_2}/\mathrm{Im}(I_{K_2})$ preserves the Frob. elt $\mathrm{Frob}_{K_i}$

(6). the collection of the isom's $\{(\alpha|_{U_1})^{ab} : U_1^{ab} \xrightarrow{\sim} U_2^{ab}\}_{U_1 \xrightarrow{\sim} U_2}$

$$\begin{array}{ccc} U_1 & \xrightarrow{\sim} & U_2 \\ \cap\,\text{open} & & \cap\,\text{open} \\ G_{K_1} & \xrightarrow[\alpha]{\sim} & G_{K_2} \end{array}$$

induces $\mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_1) \xrightarrow{\sim} \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_2)$

which is compat. w/ the actions of $G_{K_i}$ $(i=1,2)$ via $\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$

In particular, $\alpha$ preserves the cycl. char's $\chi_{cyc,i}$ $(i=1,2)$

(7) the isom $\alpha^* : H^2(\mathrm{Gal}(\overline{K}_2/K_2), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_2)) \xrightarrow{\sim} H^2(\mathrm{Gal}(\overline{K}_1/K_1), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_1))$

induced by $\alpha$ is compat. w/ the isom's $H^2(\mathrm{Gal}(\overline{K}_i/K_i), \mu_{\mathbb{Q}/\mathbb{Z}}(\overline{K}_i)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ $(i=1,2)$

$k = \overline{k}$ char $= 0$

$\mu_{\hat{\mathbb{Z}}}(k) := \varprojlim_n \mu_n(k)$

gp of $n$-th roots of $1$

$\mu_{\mathbb{Q}/\mathbb{Z}}(k) := \mu_{\hat{\mathbb{Z}}}(k) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}/\mathbb{Z}$

## §2.2 Arithmetic Quotient

**Prop 2.2** ([AbsAnab, Lem 1.1.4])

$F$: field.

$G := \mathrm{Gal}(\overline{F}/F)$ $\quad \overline{F} \supset F$ sep. closure

$$1 \to \Delta \to \Pi \to G \to 1$$

exact seq. of prof. gps

Assume $\Delta$: top. fin. gen.

(1). $F$: NF

Then $\Delta$ is gp-th'ally characterized in $\Pi$ by the max. closed normal subgp of $\Pi$ which is top. fin. gen.

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proof) Hoshi's talk.

$\uparrow$

Rem constructions are made in mono-anabelian manner

"group theoretic" algorithm

$G_1 \cong G_2$

$\rightsquigarrow$

$F(G_1) \xrightarrow{\sim} F(G_2)$

$G \rightsquigarrow F(G)$

$\overset{\shortparallel}{\mathrm{Gal}(\overline{F}/F)}$

(1).

(2).

(2). F /

Fn

Assume (

where $p$ is also gp-th'ically characterized as the unique $p \in \mathfrak{Primes}$ s.t.

$$\dim_{\mathbb{Q}_p} (\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p - \dim_{\mathbb{Q}_\ell} (\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell \neq 0$$

for $\exists \infty \ell$.

Proof) (1). $Gal(\bar{F}/F) \supset$ top. fin. gen. closed normal subgp $= \{1\}$

Fact cf. [FJ, Th 16.10]

(2). $1 \to \Delta \to \Pi \to G \to 1$

$\rightsquigarrow 1 \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(\Pi, \mathbb{Q}/\mathbb{Z}) \to H^1(\Delta, \mathbb{Q}/\mathbb{Z})^G \to \underbrace{H^2(G, \mathbb{Q}/\mathbb{Z})}_{= 0}$

$\overset{\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})}{\rightsquigarrow}$ $0 \to (\Delta^{ab})_G \to \Pi^{ab} \to G^{ab} \to 0$

$G \underset{\text{open}}{\supset} G' \rightsquigarrow F'/F$

(2). $F/\mathbb{Q}_p$ fin.

For open subgp $\Pi' \subset \Pi$, $\Delta' := \Pi' \cap \Delta$, $G' := \Pi'/\Delta'$

let $G'$ act on $(\Delta')^{ab}$ by conj.

Assume (Tam 1) $\forall \Pi' \subset \Pi$ open, $Q := \underset{\text{coinv.}}{((\Delta')^{ab})_{G'}} / (\mathrm{tor})$ is a fin. gen. free $\hat{\mathbb{Z}}$-module.

Then $\Delta$ is gp th'cally characterized in $\Pi$ as the intersection of those open subgps $\Pi' \subset \Pi$ s.t. for $\forall$ primes $\ell \neq p$

(Tam 2) $\dim_{\mathbb{Q}_p} (\Pi')^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p - \dim_{\mathbb{Q}_\ell} (\Pi')^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell \dim_{\mathbb{Q}_\ell} (\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell$

$= [\Pi : \Pi'] (\dim_{\mathbb{Q}_p} (\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p - \dim_{\mathbb{Q}_\ell} (\Pi)^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell)$.

alg. closure

s

ed in $\Pi$

normal subgp of $\Pi$

. fin. gen.

mono-anabelian manner

"group theoretic" algorithm

$G_1 \simeq G_2$

$\downarrow$

$F(G_1) \xrightarrow{\sim} F(G_2)$

$G \rightsquigarrow \underline{F(G)}$

$\overset{\shortparallel}{Gal(\bar{F}/F)}$

(6). th

(7)

(Tam

Lem 2.3 ([Abstract, Lem 1.1.5])

$F/\mathbb{Q}_p$ fin., $A$: semi-abelian var. $/F$

$\bar{F} \supset F$ alg. clsre $G := \mathrm{Gal}(\bar{F}/F)$

$T(A) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A(\bar{F}))$ Tate module of $A$

$\Longrightarrow Q := T(A)_G/(\mathrm{tor})$ is a fin. gen. free $\hat{\mathbb{Z}}$-module.

$(\mathrm{Tam}\,1) \sim \dim_{\mathbb{Q}_p}(\Pi')^{ab}\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}_p - \dim_{\mathbb{Q}_\ell}(\Pi')^{ab}\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}_\ell$

$= \dim_{\mathbb{Q}_p}(G')^{ab}\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}_p - \dim_{\mathbb{Q}_\ell}(G')^{ab}\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}_\ell \underset{\substack{\uparrow \\ \mathrm{LCFT}}}{\overset{\ell \neq p}{=}} [F':\mathbb{Q}_p]$

$\sim$ gp thic char. of $p$

$(\mathrm{Tam}\,2) \Leftrightarrow [F':\mathbb{Q}_p] = [\Pi:\Pi'][F:\mathbb{Q}_p]$

$\Leftrightarrow [\Pi:\Pi'] = [G:G']$

$\Leftrightarrow \Delta = \Delta'$ //

which is top. fin. gen.

<u>Cor</u> $X$ : geom. conn. smooth hyperb. curve $/ F /_{\mathbb{Q}}^{\text{fin.}}$

$\exists$ We have a gp th'c characterization

of $\Delta := \pi_1(X_{\overline{F}}, \overline{x})$ in $\Pi := \pi_1(X, \overline{x})$

$X_{\overline{F}} = X \otimes_F \overline{F}$

NF, MLF

$\Pi \rightsquigarrow \Delta \subset \Pi \twoheadrightarrow G$

$\mathbb{F}_p \not\cong \mathbb{F}_q$

$G_{\mathbb{F}_p} \cong G_{\mathbb{F}_q}$

$\cong \hat{\mathbb{Z}}$